

# ON FINITE $p$ -GROUPS WHOSE AUTOMORPHISMS ARE ALL CENTRAL

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**ABSTRACT.** An automorphism  $\alpha$  of a group  $G$  is said to be central if  $\alpha$  commutes with every inner automorphism of  $G$ . We construct a family of non-special finite  $p$ -groups having abelian automorphism groups. These groups provide counterexamples to a conjecture of A. Mahalanobis [Israel J. Math., **165** (2008), 161 - 187]. We also construct a family of finite  $p$ -groups having non-abelian automorphism groups and all automorphisms central. This solves a problem of I. Malinowska [Advances in group theory, Aracne Editrice, Rome 2002, 111-127].

## 1. INTRODUCTION

Let  $G$  be a finite group. An automorphism  $\alpha$  of  $G$  is called *central* if  $x^{-1}\alpha(x) \in Z(G)$  for all  $x \in G$ , where  $Z(G)$  denotes the center of  $G$ . The set of all central automorphisms of  $G$  is a normal subgroup of  $\text{Aut}(G)$ , the group of all automorphisms of  $G$ . We denote this group by  $\text{Autcent}(G)$ . Notice that  $\text{Autcent}(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$ , the centralizer of  $\text{Inn}(G)$  in  $\text{Aut}(G)$ , and  $\text{Autcent}(G) = \text{Aut}(G)$  if  $\text{Aut}(G)$  is abelian. We denote the commutator and Frattini subgroup of  $G$  with  $\gamma_2(G)$  and  $\Phi(G)$ , respectively. Let  $G^p = \langle x^p \mid x \in G \rangle$  and  $G_p = \langle x \in G \mid x^p = 1 \rangle$ . For finite abelian groups  $H$  and  $K$ ,  $\text{Hom}(H, K)$  denotes the group of all homomorphisms from  $H$  to  $K$ . Throughout the paper,  $p$  always denotes an odd prime.

In this paper we construct examples of finite  $p$ -groups whose automorphisms are all central. First we consider the case when  $\text{Aut}(G)$  is abelian for a given group  $G$ . In 1908, P. Hilton [11, p 233] asked the following question: Whether a non-abelian group can have an abelian group of isomorphisms (automorphisms). An affirmative answer to this question was given by G. A. Miller [18] in 1913. He constructed a non-abelian group  $G$  of order 64 such that  $\text{Aut}(G)$  is abelian and has order 128. More examples of such finite 2-groups were constructed by R. R.

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Struik [21] in 1982, M. J. Curran [4] in 1987 and A. Jamali [13] in 2002. In 1974, H. Heineken and H. Liebeck [9] showed that for any finite group  $K$ , there exists a finite  $p$ -group  $G$  such that  $\text{Aut}(G)/\text{Autcent}(G)$  is isomorphic to  $K$ . In particular, for  $K = 1$ , this provides a  $p$ -group  $G$  such that  $\text{Aut}(G) = \text{Autcent}(G)$  is an elementary abelian group. In 1975, D. Jonah and M. Konvisser [14] constructed 4-generated groups of order  $p^8$  such that  $\text{Aut}(G) = \text{Autcent}(G)$  and  $\text{Aut}(G)$  is an elementary abelian group of order  $p^{16}$ , where  $p$  is any prime.

In 1927, C. Hopkins [12] proved, among other things, that if  $G$  is a group such that  $\text{Aut}(G)$  is abelian, then  $G$  can not have a non-trivial abelian direct factor. But this result is not true for 2-groups, as proved by B. Earnley in his thesis [5, Theorem 2.3] in 1975. Among other things, Earnley proved (i) there is no group  $G$  of order  $p^5$  such that  $\text{Aut}(G)$  is abelian, (ii) for each positive integer  $n \geq 4$ , there exist  $n$ -generated  $p$ -groups  $G$  such that  $\text{Aut}(G)$  is abelian. On the way to constructing finite  $p$ -groups of class 2 such that all normal subgroups of  $G$  are characteristic, in 1979 H. Heineken [10] produced groups  $G$  such that  $\text{Aut}(G)$  is abelian. In 1994, M. Morigi [19] proved that there exists no group of order  $p^6$  whose group of automorphisms is abelian and constructed groups  $G$  of order  $p^{n^2+3n+3}$  such that  $\text{Aut}(G)$  is abelian, where  $n$  is a positive integer. In particular, for  $n = 1$ , it provides a group of order  $p^7$  having an abelian automorphism group. Further in 1995, M. Morigi [20] proved that the minimal number of generators for a  $p$ -group with abelian automorphism group is 4. In 1995, P. Hegarty [8] proved that if  $G$  is a non-abelian  $p$ -group such that  $\text{Aut}(G)$  is abelian, then  $|\text{Aut}(G)| \geq p^{12}$ , and the minimum is obtained by the group of order  $p^7$  constructed by M. Morigi. Moreover, in 1998 G. Ban and S. Yu [2] obtained independently the same result and proved that if  $G$  is a group of order  $p^7$  such that  $\text{Aut}(G)$  is abelian, then  $|\text{Aut}(G)| = p^{12}$  (the last result is true for all primes, not only for  $p$  odd).

We would like to remark here that all the examples mentioned above are special  $p$ -groups, where  $p$  is an odd prime. Until recently, no non-special  $p$ -group  $G$  was known (to the best of our knowledge) such that  $\text{Aut}(G)$  is abelian. Our last statement is supported by the following conjecture of A. Mahalanobis [15]:

**Conjecture.** *For an odd prime  $p$ , let  $G$  be a finite  $p$ -group such that  $\text{Aut}(G)$  is abelian. Then  $G$  is a special  $p$ -group.*

We construct a family of counterexamples to this conjecture in the following theorem, which we prove in Section 2.

**Theorem A.** *Let  $m = n + 5$  and  $p$  be an odd prime, where  $n$  is a positive integer greater than or equal to 3. Then there exists a 4-generated group  $G$  of order  $p^m$  and exponent  $p^n$  such that  $\text{Aut}(G)$  is abelian, but  $G$  is not special. Moreover,  $|\text{Aut}(G)| = p^{n+10}$ .*

Now we consider finite  $p$ -groups  $G$  for which  $\text{Aut}(G) = \text{Autcent}(G)$  is non-abelian. In 1982, M. J. Curran [3] constructed groups  $G$  of order  $2^7$  such that  $\text{Aut}(G) = \text{Autcent}(G)$  is non-abelian. Further, in 1984, J. J. Malone [17] constructed  $p$ -groups for odd primes such that  $\text{Aut}(G) = \text{Autcent}(G)$  is non-abelian. We would like to remark here that the groups of Curran and Malone have direct factors. More precisely these groups were constructed by taking direct products of abelian (cyclic)  $p$ -groups and groups  $G$  such that  $\text{Aut}(G)$  is abelian. Examples of 2-groups  $G$  such that  $G$  does not have an abelian direct factor and  $\text{Aut}(G) = \text{Autcent}(G)$  is non-abelian were constructed by S. P. Glasby [6] in 1986. Until recently, no examples of such  $p$ -groups were known (to the best of our knowledge) for an odd prime  $p$ . Our last statement is supported by the following problem of I. Malinowska [16, Problem 13].

**Problem.** *For an odd prime  $p$ , find a  $p$ -group  $G$  which has no non-trivial abelian direct factor and  $\text{Aut}(G) = \text{Autcent}(G)$  is non-abelian.*

We construct examples of such groups in the following theorem, which we prove in Section 3.

**Theorem B.** *Let  $m = n + 7$  and  $p$  be an odd prime, where  $n$  is a positive integer greater than or equal to 3. Then there exists a group  $G$  of order  $p^m$ , exponent  $p^n$  and with no non-trivial abelian direct factor such that  $\text{Aut}(G) = \text{Autcent}(G)$  is non-abelian.*

We would like to remark that before writing proofs of the above theorems, we used GAP [7] to establish the validity of these results for the following pairs  $(p, n)$ :  $\{(3, n) \mid 3 \leq n \leq 7\}$ ,  $\{(5, n) \mid 3 \leq n \leq 7\}$ ,  $\{(7, n) \mid 3 \leq n \leq 5\}$ .

2. GROUPS  $G$  WITH  $\text{Aut}(G)$  ABELIAN

In this section we construct an infinite family of finite  $p$ -groups  $G$  such that  $G$  is not a special  $p$ -group and  $\text{Aut}(G)$  is abelian, where  $p$  is an odd prime.

Let  $n$  be a natural number greater than 2 and  $p$  an odd prime. Define

$$(2.1) \quad G = \langle x_1, x_2, x_3, x_4 \mid x_1^{p^n} = x_2^{p^2} = x_3^{p^2} = x_4^p = 1, [x_1, x_2] = x_2^p, \\ [x_1, x_3] = x_3^p, [x_1, x_4] = x_3^p, [x_2, x_3] = x_1^{p^{n-1}}, [x_2, x_4] = x_2^p, \\ [x_3, x_4] = 1 \rangle.$$

Throughout this section,  $G$  always denotes the group defined in (2.1).

**Lemma 2.1.** *The group  $G$  is a regular  $p$ -group of nilpotency class 2, the exponent of  $G$  is  $p^n$  and  $Z(G) = \Phi(G)$ .*

*Proof.* By using the given relations, it is easy to show that the commutators  $[x_i, x_j] \in Z(G)$ , where  $1 \leq i < j \leq 4$ . Thus for any elements  $g_1, g_2 \in G$ ,  $[g_1, g_2]$  can be expressed as a product of powers of  $[x_i, x_j]$ ,  $1 \leq i < j \leq 4$ . This shows that  $\gamma_2(G) \subseteq Z(G)$  and therefore the nilpotency class of  $G$  is 2. Since  $p$  is odd, it follows that  $G$  is regular. From the presentation of  $G$  and the fact that  $\gamma_2(G)$  is elementary abelian it follows that the exponent of  $G$  is  $p^n$ . Since  $G^p = Z(G)$ , it follows that  $\Phi(G) = \gamma_2(G)G^p = Z(G)$ .  $\square$

The following two results are well known.

**Lemma 2.2.** *Let  $A$ ,  $B$  and  $C$  be finite abelian groups. Then  $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$  and  $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$ .*

**Lemma 2.3.** *Let  $C_r$  and  $C_s$  be two cyclic groups of order  $r$  and  $s$  respectively. Then  $\text{Hom}(C_r, C_s) \cong C_d$ , where  $d$  is the greatest common divisor of  $r$  and  $s$ .*

A group  $H$  is said to be *purely non-abelian* if it does not have a non-trivial abelian direct factor. Now we calculate the order of  $\text{Autcent}(G)$ .

**Lemma 2.4.** *Let  $G$  be the group defined in (2.1). Then  $|\text{Autcent}(G)| = p^{n+10}$ .*

*Proof.* Since  $Z(G) = \Phi(G)$ ,  $G$  is purely non-abelian. Then by Theorem 1 of [1], there is one-to-one correspondence between  $\text{Autcent}(G)$  and  $\text{Hom}(G/\gamma_2(G), Z(G))$ . Notice that  $G/\gamma_2(G) \cong C_{p^{n-1}} \times C_p \times C_p \times C_p$  and  $Z(G) \cong C_{p^{n-1}} \times C_p \times C_p$ .

Thus using Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} |\operatorname{Hom}(G/\gamma_2(G), Z(G))| &\cong |\operatorname{Hom}(C_{p^{n-1}} \times C_p \times C_p \times C_p, C_{p^{n-1}} \times C_p \times C_p)| \\ &\cong p^{n+1} p^3 p^3 p^3 = p^{n+10}. \end{aligned}$$

Hence  $|\operatorname{Autcent}(G)| = p^{n+10}$ .  $\square$

Let  $e_{x_i} = x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}} x_4^{a_{i4}} = \prod_{j=1}^4 x_j^{a_{ij}}$ , where  $x_i \in G$  and  $a_{ij}$  are non-negative integers for  $1 \leq i, j \leq 4$ . Since the nilpotency class of  $G$  is 2, we have

$$(2.2) \quad [x_k, e_{x_i}] = [x_k, \prod_{j=1}^4 x_j^{a_{ij}}] = \prod_{j=1}^4 [x_k, x_j^{a_{ij}}] = \prod_{j=1}^4 [x_k, x_j]^{a_{ij}}$$

and

$$\begin{aligned} (2.3) \quad [e_{x_k}, e_{x_i}] &= [\prod_{l=1}^4 x_l^{a_{kl}}, \prod_{j=1}^4 x_j^{a_{ij}}] = \prod_{j=1}^4 \prod_{l=1}^4 [x_l^{a_{kl}}, x_j^{a_{ij}}] \\ &= \prod_{j=1}^4 \prod_{l=1}^4 [x_l, x_j]^{a_{kl} a_{ij}}. \end{aligned}$$

Equations (2.2) and (2.3) will be used for our calculations without any further reference.

Let  $\alpha$  be an automorphism of  $G$ . Since the nilpotency class of  $G$  is 2 and  $\gamma_2(G)$  is generated by  $x_1^{p^{n-1}}$ ,  $x_2^p$ ,  $x_3^p$ , we can write  $\alpha(x_i) = x_i e_{x_i} = x_i \prod_{j=1}^4 x_j^{a_{ij}}$  for some non-negative integers  $a_{ij}$  for  $1 \leq i, j \leq 4$ .

**Proposition 2.5.** *Let  $G$  be the group defined in (2.1) and  $\alpha$  be an automorphism of  $G$  such that  $\alpha(x_i) = x_i e_{x_i} = x_i \prod_{j=1}^4 x_j^{a_{ij}}$ , where  $a_{ij}$  are some non-negative*

integers for  $1 \leq i, j \leq 4$ . Then the following equations hold:

$$\begin{aligned}
(2.4) \quad & a_{4j} \equiv 0 \pmod{p}, \quad 1 \leq j \leq 3, \\
(2.5) \quad & a_{21} \equiv 0 \pmod{p}, \\
(2.6) \quad & a_{11} + a_{11}a_{22} - a_{14} + a_{12}a_{24} - a_{14}a_{22} \equiv 0 \pmod{p}, \\
(2.7) \quad & a_{11}a_{23} + a_{24} + a_{11}a_{24} \equiv 0 \pmod{p}, \\
(2.8) \quad & a_{31} \equiv 0 \pmod{p}, \\
(2.9) \quad & a_{11} + a_{11}a_{33} + a_{34} + a_{11}a_{34} \equiv 0 \pmod{p}, \\
(2.10) \quad & a_{12} - a_{32} + a_{12}a_{44} \equiv 0 \pmod{p}, \\
(2.11) \quad & a_{44} - a_{33} + a_{11} + a_{11}a_{44} \equiv 0 \pmod{p}, \\
(2.12) \quad & a_{33} + a_{22} + a_{22}a_{33} - a_{23}a_{32} - a_{11} \equiv 0 \pmod{p}, \\
(2.13) \quad & a_{34} + a_{22}a_{34} - a_{24}a_{32} \equiv 0 \pmod{p}, \\
(2.14) \quad & a_{44} + a_{22}a_{44} \equiv 0 \pmod{p}, \\
(2.15) \quad & a_{23} \equiv 0 \pmod{p}, \\
(2.16) \quad & a_{32} + a_{32}a_{44} \equiv 0 \pmod{p}.
\end{aligned}$$

*Proof.* Let  $\alpha$  be the automorphism of  $G$  such that  $\alpha(x_i) = x_i e_{x_i}$ ,  $1 \leq i \leq 4$ . Since  $G$  is regular,  $G_p = \langle x_1^{p^{n-1}}, x_2^p, x_3^p, x_4^p \rangle$ . We know that  $G_p$  is characteristic, therefore  $\alpha(x_4) = x_4 e_{x_4} \in G_p$ . This shows that  $e_{x_4} \in G_p$ . Thus  $a_{4j} \equiv 0 \pmod{p}$  for  $1 \leq j \leq 3$ . This proves that equation (2.4) holds.

We prove equations (2.5) - (2.7) by comparing the powers of  $x_i$ 's in  $\alpha([x_1, x_2]) = \alpha(x_2^p)$ .

$$\begin{aligned}
\alpha([x_1, x_2]) &= [\alpha(x_1), \alpha(x_2)] = [x_1 e_{x_1}, x_2 e_{x_2}] \\
&= [x_1, x_2][x_1, e_{x_2}][e_{x_1}, x_2][e_{x_1}, e_{x_2}] \\
&= [x_1, x_2] \prod_{j=1}^4 [x_1, x_j]^{a_{2j}} \prod_{j=1}^4 [x_2, x_j]^{-a_{1j}} \prod_{j=1}^4 \prod_{l=1}^4 [x_l, x_j]^{a_{1l}a_{2j}} \\
&= [x_1, x_2]^{1+a_{22}+a_{11}+a_{11}a_{22}-a_{12}a_{21}} [x_1, x_3]^{a_{23}+a_{11}a_{23}-a_{13}a_{21}} \\
&\quad [x_1, x_4]^{a_{24}+a_{11}a_{24}-a_{14}a_{21}} [x_2, x_3]^{-a_{13}+a_{12}a_{23}-a_{13}a_{22}} \\
&\quad [x_2, x_4]^{-a_{14}+a_{12}a_{24}-a_{14}a_{22}} [x_3, x_4]^{a_{13}a_{24}-a_{14}a_{23}} \\
&= x_1^{p^{n-1}(-a_{13}+a_{12}a_{23}-a_{13}a_{22})} \\
&\quad x_2^{p(1+a_{22}+a_{11}+a_{11}a_{22}-a_{12}a_{21}-a_{14}+a_{12}a_{24}-a_{14}a_{22})} \\
&\quad x_3^{p(a_{23}+a_{11}a_{23}-a_{13}a_{21}+a_{24}+a_{11}a_{24}-a_{14}a_{21})}.
\end{aligned}$$

On the other hand

$$\alpha([x_1, x_2]) = \alpha(x_2^p) = x_2^p x_1^{pa_{21}} x_2^{pa_{22}} x_3^{pa_{23}} x_4^{pa_{24}} = x_1^{pa_{21}} x_2^{pa_{22}+p} x_3^{pa_{23}}.$$

Comparing the powers of  $x_1$ , we get  $a_{21} \equiv 0 \pmod{p}$ . Using this fact and comparing the powers of  $x_2$  and  $x_3$ , we get

$$a_{11} + a_{11}a_{22} - a_{14} + a_{12}a_{24} - a_{14}a_{22} \equiv 0 \pmod{p},$$

$$a_{11}a_{23} + a_{24} + a_{11}a_{24} \equiv 0 \pmod{p}.$$

Hence equations (2.5) - (2.7) hold.

Equations (2.8) and (2.9) are proved in the same way by comparing the powers of  $x_1$  and  $x_3$  in  $\alpha([x_1, x_3]) = \alpha(x_3^p)$ . Equations (2.10) and (2.11) are proved by comparing the powers of  $x_2$  and  $x_3$  in  $\alpha([x_1, x_4]) = \alpha(x_3^p)$  and using (2.4). Equations (2.12) and (2.13) are proved by comparing the powers of  $x_1$  and  $x_2$  in  $\alpha([x_2, x_3]) = \alpha(x_1^{p^{n-1}})$  and using (2.5) and (2.8). Equations (2.14) and (2.15) are proved by comparing the powers of  $x_2$  and  $x_3$  in  $\alpha([x_2, x_4]) = \alpha(x_2^p)$  and using (2.4) and (2.5). Finally, equation (2.16) is proved by comparing the powers of  $x_2$  in  $\alpha([x_3, x_4]) = 1$  and using (2.4).  $\square$

**Theorem 2.6.** *Let  $G$  be the group defined in (2.1). Then all automorphisms of  $G$  are central.*

*Proof.* Let  $\alpha$  be an automorphism of  $G$  such that  $\alpha(x_i) = x_i e_{x_i}$ , where  $e_{x_i} = \prod_{j=1}^4 x_j^{a_{ij}}$ ,  $a_{ij}$  are non-negative integers for  $1 \leq i, j \leq 4$ . To complete the proof, it is sufficient to show that  $e_{x_i} \in Z(G)$  for  $1 \leq i \leq 4$ . Since  $Z(G) = \Phi(G)$ , we shall show that  $a_{ij} \equiv 0 \pmod{p}$  for  $1 \leq i, j \leq 4$ .

We start by showing that  $a_{44} + 1$  is not divisible by  $p$ . For, if  $a_{44} + 1$  is divisible by  $p$ , then it follows from equation (2.4) that  $\alpha(x_4) \in Z(G)$ , which is not possible. Using this fact and equation (2.16), we get  $a_{32} \equiv 0 \pmod{p}$ . Thus equation (2.10) gives  $a_{12} \equiv 0 \pmod{p}$ . We claim that  $p$  does not divide  $a_{11} + 1$ . For, if  $p$  divides  $a_{11} + 1$ , then the order of  $\alpha(x_1)$  is at most  $p^{n-1}$ , which is not possible. Using (2.15) and the fact that  $1 + a_{11} \not\equiv 0 \pmod{p}$ , equation (2.7) implies  $a_{24} \equiv 0 \pmod{p}$ . Since  $a_{2i} \equiv 0 \pmod{p}$ ,  $i = 1, 3, 4$ , it follows that  $\alpha(x_2) \equiv x_2^{1+a_{22}} \pmod{Z(G)}$ . Hence  $1 + a_{22} \not\equiv 0 \pmod{p}$ .

Since  $a_{32} \equiv 0 \pmod{p}$  and  $a_{22} + 1 \not\equiv 0 \pmod{p}$ , from equations (2.13) and (2.14), we have  $a_{34}$  and  $a_{44}$  are both congruent to  $0 \pmod{p}$ . Here we claim that  $a_{33} + 1$  is not divisible by  $p$ . For, otherwise  $\alpha(x_3) \in Z(G)$ , which is not possible. Now

using the fact that  $a_{34} \equiv 0 \pmod p$ , it follows from equation (2.9) that  $a_{11} \equiv 0 \pmod p$ . Since  $a_{44} \equiv 0 \pmod p$ , from equation (2.11) we have  $a_{33} \equiv a_{11} \pmod p$ . Hence  $a_{33} \equiv 0 \pmod p$ . Since  $a_{11}, a_{12}$  are congruent to  $0 \pmod p$  and  $1 + a_{22} \not\equiv 0 \pmod p$ , it follows from equation (2.6) that  $a_{14} \equiv 0 \pmod p$ . Putting  $a_{11}, a_{23}$  and  $a_{33}$  equal to  $0 \pmod p$  in equation (2.12), we get  $a_{22} \equiv 0 \pmod p$ .

It only remains to show that  $a_{13} \equiv 0 \pmod p$ . Using above information, notice that  $e_{x_2}, e_{x_4} \in Z(G)$ . Thus

$$x_2^p = [x_2, x_4] = \alpha([x_2, x_4]) = \alpha(x_2^p) = x_1^{pa_{21}} x_2^{p(a_{22}+1)} x_3^{pa_{23}}.$$

This implies that  $x_1^{pa_{21}} = 1$ . Since  $e_{x_2}, x_1^{a_{11}}, x_2^{a_{12}}, x_4^{a_{14}} \in Z(G)$ , we get

$$\alpha([x_1, x_2]) = [x_1, x_2][x_3, x_2]^{a_{13}} = x_2^p x_1^{-p^{n-1}a_{13}}.$$

This gives

$$x_2^p x_1^{-p^{n-1}a_{13}} = \alpha([x_1, x_2]) = \alpha(x_2^p) = x_1^{pa_{21}} x_2^{p(a_{22}+1)} x_3^{pa_{23}}.$$

This, by comparing the powers of  $x_1$  and using the fact that  $x_1^{pa_{21}} = 1$ , implies  $x_1^{-p^{n-1}a_{13}} = 1$ , which in turn implies that  $a_{13} \equiv 0 \pmod p$ . Hence  $a_{ij} \equiv 0 \pmod p$  for all  $1 \leq i, j \leq 4$ . Since  $\alpha$  is an arbitrary automorphism of  $G$ , this completes the proof of the theorem.  $\square$

Let  $A$  be an abelian  $p$ -group and  $a \in A$ . For a positive integer  $n$ ,  $p^n$  is said to be the *height* of  $a$  in  $A$ , denoted by  $\text{ht}(a)$ , if  $a \in A^{p^n}$  but  $a \notin A^{p^{n+1}}$ . Let  $H$  be a  $p$ -group of class 2. We denote the exponents of  $Z(H)$ ,  $\gamma_2(H)$ ,  $H/\gamma_2(H)$  by  $p^a, p^b, p^c$  respectively and  $d = \min(a, c)$ . We define  $R := \{z \in Z(H) \mid |z| \leq p^d\}$  and  $K := \{x \in H \mid \text{ht}(x\gamma_2(H)) \geq p^b\}$ . Notice that  $K = H^{p^b}\gamma_2(H)$ . To complete the proof of Theorem A we need the following result (in our notations) of J. E. Adney and T. Yen [1, Theorem 4].

**Theorem 2.7.** *Let  $H$  be a purely non-abelian  $p$ -group of class 2,  $p$  odd, and let  $H/\gamma_2(H) = \Pi_{i=1}^n \langle x_i \gamma_2(H) \rangle$ . Then  $\text{Autcent}(H)$  is abelian if and only if*

- (i)  $R = K$ , and
- (ii) either  $d = b$  or  $d > b$  and  $R/\gamma_2(H) = \langle x_1^{p^b} \gamma_2(H) \rangle$ .

Now we are in the position to complete the proof of Theorem A stated in the introduction.

*Proof of Theorem A.* Let  $G$  be the group defined in (2.1). By Lemma 2.1, we have  $|G| = p^{n+5}$  and the exponent of  $G$  is  $p^n$ . By Theorem 2.6, we have  $\text{Aut}(G) =$



$\text{Autcent}(G)$ . Now it follows from Lemma 2.4 that  $|\text{Aut}(G)| = p^{n+10}$ . Thus to complete the proof of the theorem, it is sufficient to prove that  $\text{Autcent}(G)$  is an abelian group. Since  $Z(G) = \Phi(G)$  (by Lemma 2.1),  $G$  is purely non-abelian. The exponents of  $Z(G)$ ,  $\gamma_2(G)$  and  $G/\gamma_2(G)$  are  $p^{n-1}$ ,  $p$  and  $p^{n-1}$  respectively. Here

$$R = \{z \in Z(G) \mid |z| \leq p^{n-1}\} = Z(G)$$

and

$$K = \{x \in G \mid \text{ht}(x\gamma_2(G)) \geq p\} = G^p\gamma_2(G) = Z(G).$$

This shows that  $R = K$ . Also  $R/\gamma_2(G) = Z(G)/\gamma_2(G) = \langle x_1^p\gamma_2(G) \rangle$ . Thus all the conditions of Theorem 2.7 are now satisfied. Hence  $\text{Autcent}(G)$  is abelian. This completes the proof of the theorem.  $\square$

### 3. GROUPS $G$ SUCH THAT $\text{Aut}(G) = \text{Autcent}(G)$ IS NON-ABELIAN

In this section we construct examples of finite  $p$ -groups  $G$  such that  $\text{Aut}(G) = \text{Autcent}(G)$  is non-abelian, where  $p$  is an odd prime.

Let  $n$  be a natural number greater than 2 and  $p$  an odd prime. Define

$$(3.1) \quad G = \langle x_1, x_2, x_3, x_4 \mid x_1^{p^n} = x_2^{p^3} = x_3^{p^2} = x_4^{p^2} = 1, [x_1, x_2] = x_2^{p^2}, \\ [x_1, x_3] = x_3^p, [x_1, x_4] = x_4^p, [x_2, x_3] = x_1^{p^{n-1}}, [x_2, x_4] = x_2^{p^2}, \\ [x_3, x_4] = x_4^p \rangle.$$

This group  $G$  is a regular  $p$ -group of nilpotency class 2 having order  $p^{n+7}$  and exponent  $p^n$ . Further  $Z(G) = \Phi(G)$  and therefore  $G$  is purely non-abelian.

Let  $\alpha$  be an automorphism of  $G$ . Since the nilpotency class of  $G$  is 2 and  $\gamma_2(G)$  is generated by  $x_1^{p^{n-1}}, x_2^{p^2}, x_3^p, x_4^p$ , we can write  $\alpha(x_i) = x_i e_{x_i} = x_i \prod_{j=1}^4 x_j^{a_{ij}}$  for some non-negative integers  $a_{ij}$  for  $1 \leq i, j \leq 4$ .

**Proposition 3.1.** *Let  $G$  be the group defined in (3.1) and  $\alpha$  be an automorphism of  $G$  such that  $\alpha(x_i) = x_i e_{x_i} = x_i \prod_{j=1}^4 x_j^{a_{ij}}$ , where  $a_{ij}$  are some non-negative*

integers for  $1 \leq i, j \leq 4$ . Then the following equations hold:

$$(3.2) \quad a_{3i} \equiv 0 \pmod{p}, \quad a_{4i} \equiv 0 \pmod{p}, \quad \text{where } i = 1, 2,$$

$$(3.3) \quad a_{43} \equiv 0 \pmod{p},$$

$$(3.4) \quad 1 + a_{44} \not\equiv 0 \pmod{p},$$

$$(3.5) \quad a_{33} \equiv 0 \pmod{p},$$

$$(3.6) \quad a_{21} \equiv 0 \pmod{p},$$

$$(3.7) \quad a_{44}(1 + a_{22}) \equiv 0 \pmod{p},$$

$$(3.8) \quad a_{23} \equiv 0 \pmod{p},$$

$$(3.9) \quad a_{22} - a_{11} \equiv 0 \pmod{p},$$

$$(3.10) \quad a_{24} \equiv 0 \pmod{p},$$

$$(3.11) \quad a_{11} \equiv 0 \pmod{p},$$

$$(3.12) \quad a_{13}a_{34} - a_{14} \equiv 0 \pmod{p},$$

$$(3.13) \quad a_{13} \equiv 0 \pmod{p}.$$

*Proof.* Let  $\alpha$  be the automorphism of  $G$  such that  $\alpha(x_i) = x_i e_{x_i}$ ,  $1 \leq i \leq 4$ . Since  $G$  is regular,  $G_{p^2} = \langle x_1^{p^{n-2}}, x_2^p, x_3, x_4 \rangle$ . We know that  $G_{p^2}$  is characteristic, therefore  $\alpha(x_i) = x_i e_{x_i} \in G_{p^2}$ , where  $i = 3, 4$ . This shows that  $e_{x_i} \in G_{p^2}$  for  $i = 3, 4$ . Thus  $a_{ij} \equiv 0 \pmod{p}$  for  $i = 3, 4$  and  $j = 1, 2$ . This proves that equation (3.2) holds.

Equation (3.3) is proved by comparing the powers of  $x_3$  in  $\alpha([x_3, x_4]) = \alpha(x_4^p)$  and using (3.2). Since  $a_{4i} \equiv 0 \pmod{p}$ ,  $1 \leq i \leq 3$ , it follows that  $x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{43}} \in Z(G)$ . Suppose  $1 + a_{44} \equiv 0 \pmod{p}$ . Then  $\alpha(x_4) = x_4 e_{x_4} = x_4^{1+a_{44}} x_1^{a_{41}} x_2^{a_{42}} x_3^{a_{43}} \in Z(G)$ , which is not possible. This proves equation (3.4). Equation (3.5) is proved by comparing the powers of  $x_4$  in  $\alpha([x_3, x_4]) = \alpha(x_4^p)$  and using (3.2) - (3.4).

Equations (3.6) - (3.8) are proved by comparing the powers of  $x_1, x_2$  and  $x_4$  in  $\alpha([x_2, x_4]) = \alpha(x_2^{p^2})$  and using the equations above. Equations (3.9) and (3.10) are proved by comparing the powers of  $x_1$  and  $x_4$  in  $\alpha([x_2, x_3]) = \alpha(x_1^{p^{n-1}})$  and using the equations above. Equations (3.11), (3.12) are proved by comparing the powers of  $x_3, x_4$  in  $\alpha([x_1, x_3]) = \alpha(x_3^p)$  and using the equations above. Finally, equation (3.13) is proved by comparing the powers of  $x_4$  in  $\alpha([x_1, x_4]) = \alpha(x_4^p)$  and using the equations above.  $\square$

*Proof of Theorem B.* Let  $G$  be the group defined in (3.1). We know that  $G$  is a purely non-abelian group of order  $p^{n+7}$  and exponent  $p^n$ . Let  $\alpha$  be an automorphism of  $G$ . Then  $\alpha(x_i) = x_i \prod_{j=1}^4 x_j^{a_{ij}}$ . To prove  $\text{Aut}(G) = \text{Autcent}(G)$ , it is sufficient to show that  $e_{x_i} \in Z(G)$  for  $1 \leq i \leq 4$ . Since  $Z(G) = \Phi(G)$ , we only need to show that  $a_{ij} \equiv 0 \pmod p$  for  $1 \leq i, j \leq 4$ .

From Proposition 3.1 it is obvious that  $a_{ij}$  is congruent to 0 modulo  $p$  for  $1 \leq i, j \leq 4$ , except for  $a_{12}$  and  $a_{34}$ . Since  $e_{x_4} \in Z(G)$  and  $x_1^{a_{31}} x_2^{a_{32}} x_3^{a_{33}} \in Z(G)$ , we have  $\alpha([x_3, x_4]) = [x_3 x_4^{a_{34}}, x_4] = [x_3, x_4] = x_4^p$ . Furthermore,  $\alpha([x_1, x_4]) = [x_1 x_2^{a_{12}}, x_4] = [x_1, x_4][x_2, x_4]^{a_{12}} = x_4^p x_2^{p^2 a_{12}}$ , since  $x_1^{a_{11}} x_3^{a_{13}} x_4^{a_{14}} \in Z(G)$ . From the presentation of the group, we have  $[x_3, x_4] = [x_1, x_4]$ . Thus  $x_4^p = \alpha([x_3, x_4]) = \alpha([x_1, x_4]) = x_4^p x_2^{p^2 a_{12}}$ . Hence  $x_2^{p^2 a_{12}} = 1$ , which proves that  $a_{12} \equiv 0 \pmod p$ . Notice that  $e_{x_1}$  and  $e_{x_2}$  lie in  $Z(G)$  and  $(e_{x_1})^{p^{n-1}} = 1$ . Using this and comparing the powers of  $x_2$  in  $\alpha([x_2, x_3]) = \alpha(x_1^{p^{n-1}})$ , we get  $a_{34} \equiv 0 \pmod p$ . This proves that  $\text{Aut}(G) = \text{Autcent}(G)$ .

Now we proceed to show that  $\text{Autcent}(G)$  is non-abelian. The exponents of  $Z(G)$ ,  $\gamma_2(G)$  and  $G/\gamma_2(G)$  are  $p^{n-1}$ ,  $p$  and  $p^{n-1}$  respectively. Here

$$R = \{z \in Z(G) \mid |z| \leq p^{n-1}\} = Z(G)$$

and

$$K = \{x \in G \mid \text{ht}(x\gamma_2(G)) \geq p\} = G^p \gamma_2(G) = Z(G).$$

This shows that  $R = K$ . Now  $R/\gamma_2(G) = Z(G)/\gamma_2(G) = \langle x_1^p \gamma_2(G) \rangle \times \langle x_2^p \gamma_2(G) \rangle$ . This shows that condition (ii) of Theorem 2.7 is not satisfied. Hence  $\text{Autcent}(G)$  is non-abelian. This completes the proof of the theorem.  $\square$

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